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Hence, by the rules of recurring series, the sides of the $n$th triangle are the coefficients of $r^n$, $s^n$, $t^n$ in the expansion of

$$\frac{3r-r^2}{(1-r)(1-6r+r^2)},\ \frac{4s-7s^2+s^3}{(1-s)(1-6s+s^2)}\ \text{and}\ \frac{5t-t^2}{1-6t+t^2}$$

respectively.

## SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.

BY NEWTON FITZ, NORFOLK, VA.

ASSUME the identity

$x^4+ax^3+bx^2+cx+d=[x^2+(\frac{1}{2}a+\sqrt{q})x+p+r\sqrt{q}]\,[x^2+(\frac{1}{2}a-\sqrt{q})x+p-r\sqrt{q}]$. By multiplication we obtain

$x^4+ax^3+bx^2+cx+d=x^4+ax^3+(2p+\frac{1}{4}a^2-q)x^2+(ap-2rq)x+p^2-r^2q$,

whence $b=2p+\frac{1}{4}a^2-q$, $\quad c=ap-2rq$, $\quad d=p^2-r^2q$.

Eliminating $r$, and $q$, we have

$$p^3-\tfrac{1}{2}bp^2+\tfrac{1}{4}(ac-4d)p+\tfrac{1}{8}(4bd-a^2d-c^2)=0.$$

Either root of this cubic may be used in the following expressions for the roots of the biquadratic

$$\begin{aligned}
x' &= -\tfrac{1}{4}a+\tfrac{1}{2}\sqrt{q}+\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p+(a-4r)\sqrt{q}]},\\
x'' &= -\tfrac{1}{4}a+\tfrac{1}{2}\sqrt{q}-\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p+(a-4r)\sqrt{q}]},\\
x''' &= -\tfrac{1}{4}a-\tfrac{1}{2}\sqrt{q}+\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p-(a-4r)\sqrt{q}]},\\
x'''' &= -\tfrac{1}{4}a-\tfrac{1}{2}\sqrt{q}-\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p-(a-4r)\sqrt{q}]}.
\end{aligned}$$

Substituting for $r$ and $q$, their values in terms of $p$, $a$, $b$, $c$, and $d$, gives

$$x=-\frac{a}{4}\pm\frac{1}{4}\sqrt{\left(8p+a^2-4b\right)}\pm\frac{1}{2}\sqrt{\left(\frac{a^2}{2}-2p-b\pm\frac{4ap+a^3-4ab+4c}{\sqrt{(8p+a^2-4b)}}\right)}$$

As $p$ has, in general, three different values, there will be twelve different expressions for the roots of the biquadratic but only four different values, consequently there must exist relations of identity and of symmetry as follows: $p_1, p_2, p_3$ being the roots of the cubic, and $f_1, f_2, f_3, f_4$ representing the four forms of the roots of the biquadratic,

$$\begin{aligned}
x' &= f_1p_1=f_3p_2=f_4p_3, & x'' &= f_2p_1=f_4p_2=f_3p_3,\\
x''' &= f_3p_1=f_1p_2=f_2p_3, & x'''' &= f_4p_1=f_2p_2=f_1p_3.
\end{aligned}$$

The general expression for the root of an equation of the $n$th degree will contain $n$ variables, and by substituting for each of these $n$ variables conjugate quadratic surds having the same radical part we obtain the expression for the roots of an equation whose degree is $2n$, as for example,

1st degree $x = a$,
2nd " $x = \frac{1}{2}a \pm \sqrt{b}$,
4th " $x = \frac{1}{4}a \pm \sqrt{b} \pm \sqrt{(c \pm d\sqrt{b})}$,
8th " $x = \frac{1}{8}a \pm \sqrt{b} \pm \sqrt{(c \pm d\sqrt{b})} \pm \sqrt{[e \pm f\sqrt{b} \pm \sqrt{(g \pm h\sqrt{b})}]}$
3rd " $x = \frac{1}{3}a + b$ or $\frac{1}{3}a - \frac{1}{2}b \pm \sqrt{c}$,
6th " $x = \frac{1}{6}a + \frac{1}{2}b \pm \sqrt{c}$ or $\frac{1}{6}a - \frac{1}{4}b \pm \sqrt{d} \pm \sqrt{(e \pm f\sqrt{d})}$.
5th " $x = \frac{1}{5}a + b$ or $\frac{1}{5}a - \frac{1}{4}b \pm \sqrt{[c \pm \sqrt{d} \pm \sqrt{(e \pm f\sqrt{d})}]}$,
10th " $x = \frac{1}{10}a +$ &c.

To solve an equation of any degree, we form an equation of the required degree from the roots as given above and equate the coefficients thus formed to those of the given equation, and we shall have $n$ equations containing $n$ unknown quantities. Eliminating between these, the solution of the final equation containing one unknown quantity will give the solution of the original equation. The degree of the final equation may generally be determined by *a priori* considerations. Thus, if $n$ is odd, the degree of the final equation will be $n$ also, and no progress has been made; but when $n$ is any power of 2 the forms of the $n$ roots are symmetrical and we may reasonably expect some analogy in their solutions. The solution of the biquabratic is known to depend on that of the cubic, and similarly that of the *octic* should depend upon that of the *quintic*. This would give relations of identity between the roots as follows: $r$, $r'$, $r''$, &c., being roots of the octic and $q'$, $q''$, &c., roots of the corresponding quintic,

$$r' = f_1q' = f_5q'' = f_6q''' = f_7q'''' = f_8q''''',$$
$$r'' = f_2q' = f_6q'' = f_7q''' = f_8q'''' = f_5q''''',$$
$$r''' = \text{\&c.}$$

There being 8 root forms, and the five values of $q$ entering into each, there would be 40 different roots, unless the identities above written are true.

# *ON THE SUM OF THE CUBES OF ANY NUMBER OF TERMS OF ANY ARITHMETICAL SERIES.*

BY L. P. SHIDY, U. S. COAST SURVEY.

THE following propositions are thought to be original, but mathematicians may be already familiar with them.

I. The sum of the cubes of any number of consecutive terms of any arithmetical series is divisible by the sum of that series.